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On Structured Singular Values

James Demmel

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Department of Computer Science
Courant Institute of Mathematical Sciences
251 MERCER STREET, NEW YORK, N.Y. 10012

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James Demmel
Courant Institute
251 Mercer Str.
New York, NY 10012

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Abstract

Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a square, nonsingular matrix. We give explicit bounds, accurate to within a factor of at most $3^{3/2}$, for the distance from T to the nearest singular matrix where only 1, 2 or 3 of T's subblocks may be perturbed. The bounds are in terms of the norms of subblocks of T^{-1}. These results hold for a large class of matrix norms, and we present an inexpensive algorithm for estimating these bounds for the 1-norm and infinity-norm without explicitly computing T^{-1}. We extend some of these results to block 3 by 3 matrices. These results have application in H^∞ control theory and stability analysis of various problems in linear algebra.

1. Introduction

Let

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

be a square nonsingular matrix. We are interested in finding the nearest singular matrix to T where only some of the subblocks A, B, C and D may be perturbed. Our answers will be expressed in terms of the norms of the blocks of

$$T^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \tag{1.2}$$

where T^{-T} and T are conformally partitioned (i.e. A and W^T have the same dimensions, C and X^T have the same dimensions, and so on). In particular, we are interested in the following three problems:

1) What is the smallest $||\delta D||$ such that

$$\begin{bmatrix} A & C \\ B & D+\delta D \end{bmatrix}$$

is singular? The answer will turn out to be

$$\min \|\delta D\| = \frac{1}{\|Z\|} \;;$$

$\|\cdot\|$ may be an arbitrary operator norm $\|Z\| = \sup_{x \neq 0} \|Zx\|/\|x\|$. Note that $\|Z\|$ may be zero; in this case no δD makes the matrix singular. By permuting rows and columns, we see that the solution to this problem includes the cases where any other single block alone (A or B or C) is to be perturbed, as well as the case of the smallest $\|\delta D\|$ such that $[C \,|D + \delta D]$ is singular (let A, B, W and Y be null matrices). When A, B and C are null so that $T = D$ and $T^{-1} = Z$, this result reduces to the well-known result that the distance from T to the nearest singular matrix is $\|T^{-1}\|^{-1}$.

2) What is the smallest $\|\mathrm{diag}(\delta A, \delta D)\|$ such that

$$\begin{bmatrix} A + \delta A & C \\ B & D + \delta D \end{bmatrix}$$

is singular? We will show

$$\frac{.5}{\kappa} \le \min \|\mathrm{diag}(\delta A, \delta D)\| \le \frac{3^{1/2}}{\kappa}$$

where

$$\kappa = \max\,[\|Z\|\,,\,\|W\|\,,\,(\|X\|\cdot\|Y\|)^{1/2}]\ .$$

where $\|\cdot\|$ may belong to a large class of norms including all p-norms. When T is real, we prove a slightly stronger result where the upper and lower bounds can differ by a factor of at most 2. By permuting rows or columns, we see this solution also solves the case where B and C are perturbed, and A and D fixed. We also consider a related problem arising in the equality constrained quadratic programming problem, where both T and its perturbation must be symmetric.

3) What is the smallest $\left\|\begin{bmatrix} \delta A & \delta B \\ 0 & \delta D \end{bmatrix}\right\|$ such that

$$\begin{bmatrix} A + \delta A & B + \delta B \\ C & D + \delta D \end{bmatrix}$$

is singular? We will show

$$\frac{1}{3\kappa} \le \min \|\mathrm{diag}(\delta A, \delta D)\| \le \frac{3^{1/2}}{\kappa}$$

where

$$\kappa = \max\,[\|Z\|\,,\,\|W\|\,,\,\|Y\|\,,\,(\|X\|\cdot\|Y\|)^{1/2}]\ .$$

where $\|\cdot\|$ belongs to another class of norms, again including all p-norms. When T is real, we prove a slightly stronger result where the upper and lower bounds differ by a factor of 3. Permuting rows and columns, we see this solution also solves the cases where any other subblock (not just C) of T is fixed and the others vary.

The solutions to problems 2 and 3 extend immediately to perturbations

$$\begin{bmatrix} T_{11}+\delta T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22}+\delta T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{11}+\delta T_{11} & T_{12}+\delta T_{12} & T_{13} \\ T_{21} & T_{22}+\delta T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

and their block permutations. The problem where all the diagonal blocks T_{ii} may be perturbed is harder, and our upper and lower bounds may sometimes be far apart. We discuss these extensions in section 6.

Such structured perturbation problems are of interest in several areas. First, they arise in the design of control systems based on H^∞ optimization [Doyle, Fan and Tits]. Second, they solve certain constrained total least squares problems where some variables are fixed and others may vary [Demmel87b, Golub, Hoffman and Stewart]. Third, they provide a measure of stability for the equality constrained quadratic programming problem [Gould].

Even though we do not determine the smallest perturbations exactly, we do exhibit low rank perturbations which satisfy all the upper bounds given above. These may serve as starting points for optimization routines which could compute the smallest perturbation to higher accuracy.

If T is large, it may be impractical to compute T^{-1} explicitly in order to compute the norms of its sublocks. We present a method, based on a condition number estimator of Hager [Hager, Higham], for estimating the 1-norm or ∞-norm of any subblock of T^{-1} given the ability to solve $Tx=b$ and $T^Ty=c$ efficiently. Thus, for example, if T is large and sparse with a sparse LU factorization, it is possible to compute the estimates in this paper cheaply.

All these results use a technique originally introduced in [Demmel87a] for estimating the distance to the nearest "ill-posed" numerical analysis problem, such as the distance to the nearest singular matrix for the matrix inversion problem, the distance to the nearest defective matrix for the matrix eigenproblem, the distance to the nearest polynomial with multiple roots for the polynomial root finding problem, the distance to the nearest uncontrollable system for the pole placement problem, etc. In this technique we associate a *condition number* κ with each problem, which measures the sensitivity of the solution. Then we prove one or both of the following differential inequalities:

$$m\cdot\kappa^2 \le \|\nabla\kappa\| \le M\cdot\kappa^2 \tag{1.3}$$

where $\nabla\kappa$ is the gradient of κ, and $0<m\le M$ are constants. Inequalities (1.3) are equivalent to the statement that the condition number of the condition number is (nearly) itself. In [Demmel87a] it was shown the lower bound on $\|\nabla\kappa\|$ in (1.3) yields the following upper bound on the distance to the nearest ill-posed problem:

$$\text{dist} \le \frac{1}{m\cdot\kappa} \tag{1.4}$$

and the upper bound on $\|\nabla\kappa\|$ in (1.3) yields the following upper bound on the distance:

$$\frac{1}{M\cdot\kappa} \le \text{dist} \ . \tag{1.5}$$

For example, for the matrix inversion problem we choose $\kappa=\|T^{-1}\|$, in which case (1.3) is satisfied with $m=M=1$; from (1.4) and (1.5) this yields the well known result that the distance from T to the nearest singular matrix is $\|T^{-1}\|^{-1}$. We solve problems 1) to 3) above by choosing κ to satisfy one or both of the inequalities in (1.3).

For more complicated block perturbations than those in 1)-3), the inequalities in (1.3) may no longer provide explicit solutions, but they do provide a numerical optimization technique: steepest ascent on the condition number. By perturbing T in an allowed direction so as to make the norm of T^{-1} increase as quickly as possible, we can find good upper bounds on the distance to the nearest singular matrix. We intend to pursue this approach in future work.

The rest of this paper is organized as follows. Section 2 states the lemmas from [Demmel87a] which justify conclusions (1.4) and (1.5) above. Sections 3, 4 and 5 solve problem 1), 2) and 3), respectively. Section 6 discusses the block 3 by 3 matrix case. Section 7 discusses inexpensive estimators for the bounds in previous sections. Section 8 discusses future work.

2. Basic Differential Inequalities

In this section we state the basic lemmas we will need for our lower and upper bounds on the norm of the smallest rank lowering perturbation; these originally appeared in [Demmel87a]. Lemma 1 will be used to provide lower bounds on the distance to the nearest singular matrix:

Lemma 1: Let $\kappa(x)$ be a positive functional on a normed linear space $\mathcal{X}$ with norm $\|\cdot\|$. Suppose

$$\limsup_{e\to 0} \frac{|\kappa(x+e)-\kappa(x)|}{\|e\|} \le M\cdot\kappa^2(x) \qquad (2.1)$$

for all $x\in\mathcal{X}$, where M is a positive constant. Then for all x, $\kappa(z)$ is finite for all z such that

$$\|z-x\| < \frac{1}{M\cdot\kappa(x)} \quad .$$

Proof: Let $x(s)$ be a straight line in $\mathcal{X}$ starting at $x(0)=x$, and parameterized by arclength s (i.e. $\|x'(s)\|=1$). Consider the scalar function $\kappa(x(s))$. By assumption

$$\limsup_{h\to 0+} \frac{\kappa(x(s+h))-\kappa(x(s))}{h} \le M\cdot\kappa^2(x(s))$$

so that $\kappa(x(s))$ is bounded above by the function $y(s)$ where

$$y'(s) = M\cdot y^2(s) \quad , \quad y(0)=\kappa(x) \quad .$$

It is easy to verify that

$$y(s) = \frac{\kappa(x)}{1 - M\kappa(x)s}$$

which is bounded for $s < (M\kappa(x))^{-1}$. Since $x(s)$ was an arbitrary straight line starting at x, this proves $\kappa(z)$ is bounded in a ball around x of radius at least $(M\kappa(x))^{-1}$ as desired. □

If $\mathcal{X}$ is Euclidean space and κ is differentiable, (2.1) is equivalent to $\|\nabla\kappa\| \le M\cdot\kappa^2$.

Lemma 2 will be used to provide upper bounds on the distance to the nearest singular matrix:

Lemma 2: Let $\kappa(x)$ be a positive functional on a normed linear space $\mathcal{X}$ with norm $\|\cdot\|$. Let $x(s)$ be a curve in $\mathcal{X}$ starting at $x(0)=x$ and parameterized by arclength s. Suppose

$$\liminf_{h\to 0+} \frac{\kappa(x(s+h))-\kappa(x(s))}{h} \ge m\cdot\kappa^2(x(s)) \qquad (2.2)$$

for all s, where m is a positive constant. Then $\kappa(z)$ is infinite for some z satisfying

$$\|x-z\| \le \frac{1}{m\cdot\kappa(x)} \quad .$$

Proof: From (2.2) $\kappa(x(s))$ is bounded below by $y(s)$ where

$$y'(s) = m\cdot y^2(s) \quad , \quad y(0)=\kappa(x) \quad .$$

As in Lemma 1, we see

$$y(s) = \frac{\kappa(x)}{1 - m\kappa(x)s}$$

which has a pole at $s=(m\kappa(x))^{-1}$. Since $y(s)$ bounds $\kappa(x(s))$ below, $\kappa(x(s))$ must have a pole for some $s_0 \le (m\kappa(x))^{-1}$. Since s is arclength, this proves the lemma. □

If $\mathcal{X}$ is Euclidean space, κ is continuously differentiable, and $\|\cdot\|$ is the 2-norm, then assumption (2.2) may be replaced by $\|\nabla\kappa\| \ge m\kappa^2$, because this implies the curve $x(s)$ of steepest ascent of κ satisfies $\kappa'(x(s)) \ge m\cdot\kappa^2(x(s))$.

3. The Smallest Rank Lowering Perturbation of One Subblock

In this section $\|\cdot\|$ will denote any matrix operator norm, as well as the vector norm which induces it; $\|\cdot\|_D$ will denote the dual vector norm. Let T and T^{-1} be as in (1.1) and (1.2), and let $\zeta = \|Z\|$.

Theorem 1: The smallest $\|\delta D\|$ such that

$$\begin{bmatrix} A & C \\ B & D+\delta D \end{bmatrix} \tag{3.1}$$

is singular is $1/\zeta$. If $u^T Z v = \zeta$, where $\|u^T\|_D = \|v\| = 1$, then we may take $\delta D = -vu^T/\zeta$. This theorem applies to real perturbations of real matrices as well as complex perturbations of complex matrices. If the original matrix is real symmetric (complex Hermitian) and the norm is the 2-norm $\|\cdot\|_2$, the minimizing δD can be chosen real symmetric (complex Hermitian).

Proof: If δT is a small perturbation of T, then to first order $(T+\delta T)^{-1} = T^{-1} - T^{-1}\delta T T^{-1}$. Applying this to (3.1) shows that to first order in δD the inverse of the matrix in (3.1) is given by

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} - \begin{bmatrix} X\delta D Y & X\delta D Z \\ Z\delta D Y & Z\delta D Z \end{bmatrix} . \tag{3.2}$$

Consider the (2,2) entry: to first order Z is perturbed by $-Z\delta D Z$. This implies that to first order the perturbation $\delta\zeta$ in the norm ζ of Z cannot be more than $\|Z\delta D Z\| \le \zeta^2 \|\delta D\|$. Thus, by Lemma 1, the smallest $\|\delta D\|$ such that ζ is infinite satisfies

$$\min \|\delta D\| \ge \zeta^{-1} . \tag{3.3}$$

By (3.2), no other subblock of T^{-1} can become unbounded unless Z becomes unbounded, so (3.3) is also a lower bound on the smallest $\|\delta D\|$ such that the matrix in (3.1) is singular. To see that this lower bound is attained for some δD, simply choose δD proportional to the rank-one matrix $-vu^T$, where v and u^T are unit vectors satisfying $\|Z\| = u^T Z v$ (i.e. $\|u^T\|_D = \|v\| = 1$). This rank-one choice of δD perturbs Z by a rank-one matrix proportional to Zvu^TZ, which is easily seen to attain the lower bound in (3.3). Therefore, by Lemma 2, we have $\min\|\delta D\| \le \zeta^{-1}$. That $\delta D = -vu^T/\zeta$ makes T singular may also be verified via the Sherman-Morrison formula [Golub and Van Loan, p. 3].

If T is real symmetric (complex Hermitian) and the norm is the 2-norm, then Z is real symmetric (complex Hermitian) and u and v may clearly be chosen equal (complex conjugate), so that δD is real symmetric (complex Hermitian). □

Since the smallest perturbation is rank-one, we see that the smallest distance in the matrix 2-norm is also the smallest distance in the Frobenius norm; these results were obtained by other methods in [Demmel87b], where rectangular matrices of arbitrary rank were treated as well. It was also solved in [Watson].

Note that when $\zeta = 0$, no perturbation δD of D will make T singular. This implies that the distance $1/\zeta$ may be a very ill-conditioned function of the entries of T. For example, if

$$T = \begin{bmatrix} \epsilon & 1 \\ 1 & 0 \end{bmatrix}$$

then the smallest perturbation of the 2,2 entry which makes T singular is ϵ^{-1}. As ϵ goes to zero, the norm of this perturbation goes to infinity, and when $\epsilon = 0$, no perturbation lowers the rank. In the case of the 2-norm, a sensitivity analysis of $1/\zeta$ was done in [Demmel87b].

When A is square and nonsingular, a more elementary construction yields the same result:

$$\begin{bmatrix} A & B \\ C & D+\delta D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} A & B \\ 0 & D-CA^{-1}B+\delta D \end{bmatrix}$$

is singular if and only if $D-CA^{-1}B+\delta D$ is singular, and the smallest perturbation $\|\delta D\|$ that makes this block singular is given by the well known expression $\|(D-CA^{-1}B)^{-1}\|^{-1}$. But

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

so $\|Z\| = \|(D-BA^{-1}C)^{-1}\|$ as desired.

4. The Smallest Rank Lowering Perturbation of Two Sublocks

In this section we are interested in the smallest $\|\delta T\|$ such that $T+\delta T$ is singular where

$$\delta T = \begin{bmatrix} \delta A & 0 \\ 0 & \delta D \end{bmatrix} .$$

Here $\|\cdot\|$ may be any operator norm subject to the restriction

$$\max\,(\|\delta A\|\ ,\ \|\delta D\|) = \left\| \begin{bmatrix} \delta A & 0 \\ 0 & \delta D \end{bmatrix} \right\| ;$$

this is satisfied, for example, by all the matrix p-norms. As before, $\|\cdot\|$ will also denote the vector norm which induces the matrix norm, and $\|\cdot\|_D$ the dual norm, and

$$T^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} .$$

Let $\omega=\|W\|$, $\xi=\|X\|$, $\eta=\|Y\|$, and $\zeta=\|Z\|$.

Theorem 2: Let $\kappa = \max(\zeta,\omega,(\xi\eta)^{1/2})$. Then the smallest $\|\mathrm{diag}(\delta A,\delta B)\|$ such that

$$\begin{bmatrix} A+\delta A & B \\ C & D+\delta D \end{bmatrix}$$

is singular is bounded by

$$\frac{1}{2\kappa} \le \frac{1}{\max(\omega,\zeta)\ +\ (\xi\eta)^{1/2}} \le \min\,\|\mathrm{diag}(\delta A,\delta B)\| \le \frac{\gamma}{\kappa} ,$$

where $\gamma \le 3^{1/2}$. More specifically, if we define $\alpha = \max(\omega,\zeta)\ /\ (\xi\eta)^{1/2}$ then

$$\gamma = \begin{cases} 1 & \text{if } \alpha \ge 1 \\ \dfrac{1}{\alpha} & \text{if } 3^{-1/2} < \alpha < 1 \\ \dfrac{(1+\alpha^2)^{1/2}}{1-\alpha^2} & \text{if } \alpha \le 3^{-1/2} \end{cases} \tag{4.1}$$

When T and δT are real, this result can be improved to

$$\frac{1}{\max(\omega,\zeta)\ +\ (\xi\eta)^{1/2}} \le \min\,\|\mathrm{diag}(\delta A,\delta B)\| \le \frac{1}{\max(\omega,\zeta,(\xi\eta)^{1/2})} ;$$

note that the upper bound is at most twice the lower bound.

Proof: To first order,

$$(T+\delta T)^{-1} = T^{-1} - T^{-1}\delta T T^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} - \begin{bmatrix} W\delta AW + X\delta DY & W\delta AX + X\delta DZ \\ Y\delta AW + Z\delta DY & Y\delta AX + Z\delta DZ \end{bmatrix}$$

Thus, to first order the perturbations in ω, ξ, η and ζ are bounded by $(\omega^2+\xi\eta)\|\delta T\|$, $\xi(\omega+\zeta)\|\delta T\|$, $\eta(\omega+\zeta)\|\delta T\|$, and $(\zeta^2+\xi\eta)\|\delta T\|$. Let $\kappa_0 = \max(\omega,\zeta) + (\xi\eta)^{1/2}$. Therefore, to first order the perturbation to κ_0 is bounded by

$$\delta\kappa_0 \le (\max(\zeta^2+\xi\eta\ ,\ \omega^2+\xi\eta) + (\omega+\zeta)(\xi\eta)^{1/2})\|\delta T\| \le \kappa_0^2\ \|\delta T\| \ ,$$

as is easily verified by computation. Note that $\|T^{-1}\|$ cannot become infinite unless κ_0 becomes infinite. Applying Lemma 1 as in the proof of Theorem 1, we see that the smallest $\|\delta T\| = \|\mathrm{diag}(\delta A,\delta D)\|$ such that $T+\delta T$ is singular is bounded by

$$\|\delta T\| \ge \frac{1}{\kappa_0}$$

proving the lower bound.

To prove the upper bound we will exhibit perturbations $\mathrm{diag}(\delta A,\delta D)$ which attain it. There are 3 cases as indicated in (4.1). In case 1, when $\alpha = \max(\omega,\zeta)\ /\ (\xi\eta)^{1/2} \ge 1$, we know from Theorem 1 that there is a perturbation of A alone (if $\omega \ge \zeta$) or of D alone (if $\zeta \ge \omega$) of norm at most $1/\kappa = 1/\max(\omega,\zeta)$ which makes $T+\delta T$ singular. In case 2, when $1 > \alpha > 3^{-1/2}$, the same perturbation of A alone or D alone has norm $1/\max(\omega,\zeta) = \dfrac{1}{\alpha\kappa}$. From Theorem 1, δA and δD may be chosen real symmetric (complex Hermitian) if T is real symmetric (complex Hermitian) and the norm is the 2-norm.

In case 3, when $\alpha \le 3^{-1/2}$, we must have both δA and δD nonzero. We will exhibit rank one matrices $\delta A = v_A u_A^T$ and $\delta D = v_D u_D^T$ of norms at most $(1+\alpha^2)^{1/2}/(1-\alpha^2)$ which make $T+\delta T$ singular. Let

$$V = \begin{bmatrix} v_A & 0 \\ 0 & v_D \end{bmatrix} \quad \text{and} \quad U^T = \begin{bmatrix} u_A^T & 0 \\ 0 & u_D^T \end{bmatrix}$$

be n by 2 and 2 by n matrices respectively; note that $\delta T = \mathrm{diag}(\delta A,\delta D) = VU^T$. By the Sherman-Morrison-Woodbury formula [Golub and Van Loan, p. 3] $T+\delta T$ is singular if and only if $I + U^T T^{-1} V$ is singular, i.e.

$$(1+u_A^T W v_A)\cdot(1+u_D^T Z v_D) - (u_A^T X v_D)\cdot(u_D^T Y v_A) = 0 \ . \tag{4.2}$$

We choose v_A, u_A^T, v_D, and u_D^T as follows. First find unit vectors v_ξ and u_ξ^T such that $u_\xi^T X v_\xi = \xi$, and unit vectors v_η and u_η^T such that $u_\eta^T Y v_\eta = \eta$. Now let

$$\begin{aligned} u_A^T &= u_\xi^T\ /\ (\xi\eta)^{1/4} \\ v_A &= \rho\cdot v_\eta\ /\ (\xi\eta)^{1/4} \\ u_D^T &= u_\eta^T\ /\ (\xi\eta)^{1/4} \\ v_D &= \rho\cdot v_\xi\ /\ (\xi\eta)^{1/4} \end{aligned} \quad ;$$

note that $\|\delta T\| = |\rho|/(\xi\eta)^{1/2}$ and

$$\begin{aligned} u_A^T W v_A &= \rho\cdot\underline{\omega} \quad \text{where} \quad |\underline{\omega}| \le \alpha \\ u_D^T Z v_D &= \rho\cdot\underline{\zeta} \quad \text{where} \quad |\underline{\zeta}| \le \alpha \\ (u_A^T X v_D)\cdot(u_D^T Y v_A) &= \rho^2 \end{aligned}$$

so we may rewrite (4.2) as

$$(1+\rho\cdot\underline{\omega})\cdot(1+\rho\cdot\underline{\zeta}) - \rho^2 = 0$$

whose solutions are

$$\rho_\pm = \frac{\underline{\zeta}+\underline{\omega} \pm ((\underline{\omega}-\underline{\zeta})^2 + 4)^{1/2}}{2(1 - \underline{\omega}\underline{\zeta})} \quad .$$

It is easy to verify that

$$\min |\rho_\pm| \le \frac{(1+\alpha^2)^{1/2}}{1-\alpha^2}$$

as desired. It is also easy to verify that if $0<\alpha\le 3^{-1/2}$, $(1+\alpha^2)^{1/2}/(1-\alpha^2) \le \alpha^{-1}$, and if $3^{-1/2}\le\alpha<1$, $(1+\alpha^2)^{1/2}/(1-\alpha^2) \ge \alpha^{-1}$. When T is real, $\underline{\zeta}$ and $\underline{\omega}$ are real and one may verify that $\min |\rho_\pm| \le 1$ for all $\underline{\zeta}$ and $\underline{\omega}$ less than 1 in magnitude. This completes the proof. □

The factor γ in the upper bound could be improved slightly by a more careful analysis of $\min |\rho_\pm|$; even then, for a given T, the upper bound constructed in the theorem may be smaller than the upper bound γ/κ. Note that the lower bound for the 2-norm is necessarily a lower bound for the Frobenius norm. Also, $2^{1/2}$ times the upper bound is an upper bound for the Frobenius norm because the perturbations attaining it have rank at most 2.

When T is real symmetric (complex Hermitian) and the norm is the 2-norm, then of course a solution is $\delta T = -\lambda_{\min} I$, $\|\delta T\| = |\lambda_{\min}|$, where $\lambda_{\min}$ is the eigenvalue of T of smallest absolute value. However, the Frobenius norm of this δT is $n^{1/2}$ times larger. We may construct a rank lowering symmetric perturbation whose Frobenius norm is within $2^{1/2}$ of the upper bound γ/κ for the 2-norm as follows: Since $X=Y^T$ $(X=Y^*)$ we may take $v_\xi=u_\eta$ $(v_\xi=\bar{u}_\eta)$ and $u_\xi=v_\eta$ $(u_\xi=\bar{v}_\eta)$ in the proof. The scalar ρ will be real, and both δA and δD will be rank one real symmetric (complex Hermitian), so the Frobenius norm $\|\mathrm{diag}(\delta A,\delta D)\|_F \le 2^{1/2}\|\mathrm{diag}(\delta A,\delta D)\|_2$.

A related problem to the one solved in Theorem 1 is to find the smallest $\|\delta A\|$ such that the symmetric matrix

$$\begin{bmatrix} H & A^T+\delta A^T \\ A+\delta A & M \end{bmatrix} \tag{4.3}$$

is singular. This problem arises in analyzing the stability of the equality constrained quadratic programming problem [Gould]

$$\min_{x:\, Ax=b} \frac{1}{2} x^T H x + c^T x \quad ;$$

we assume all matrices are real and use the 2-norm $\|\cdot\|_2$. The Kuhn-Tucker first order conditions for this problem are

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ -\mu \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix} \tag{4.4}$$

(μ is the vector of Lagrange multipliers).

The smallest $\|\delta A\|_2$ which makes this linear system singular is a measure of its stability, just as the usual condition number $\|A\|\cdot\|A^{-1}\|$ for $Ax=b$ is inversely proportional to the distance $\|A^{-1}\|^{-1}$ from A to the nearest singular matrix [Kahan, Demmel87a].

The problem in (4.3) is more difficult than the one solved in Theorem 2. For example, if both H and $M-AH^{-1}A^T$ are invertible, then

$$T^{-1} = \begin{bmatrix} H & A^T \\ A & M \end{bmatrix}^{-1} = \begin{bmatrix} H^{-1}+H^{-1}A^T(M-AH^{-1}A^T)^{-1}AH^{-1} & -H^{-1}A^T(M-AH^{-1}A^T)^{-1} \\ -(M-AH^{-1}A^T)^{-1}AH^{-1} & (M-AH^{-1}A^T)^{-1} \end{bmatrix} .$$

Thus, if H is positive definite and M negative definite (or vice versa), T^{-1} exists for all A. Therefore a necessary (and clearly sufficient) condition for the existence of a rank lowering perturbation δA is that both H and M have nonnegative (or nonpositive) eigenvalues; this condition cannot be captured in terms of norms of the subblocks of T^{-1}. This necessarily limits the applicability of the following theorem:

Theorem 3: Let

$$T = \begin{bmatrix} H & A^T \\ A & M \end{bmatrix} \ , \ T^{-1} = \begin{bmatrix} W & X^T \\ X & Y \end{bmatrix} \ \text{and} \ \delta T = \begin{bmatrix} 0 & \delta A^T \\ \delta A & 0 \end{bmatrix}$$

where all matrices are real and T and δT are symmetric. Let $\omega = \|W\|_2$, $\xi = \|X\|_2$, $\eta = \|Y\|_2$. The smallest $\|\delta T\|_2 = \|\delta A\|_2$ such that $T + \delta T$ is singular satisfies

$$\frac{1}{\xi + (\omega\eta)^{1/2}} \le \min \|\delta A\|_2 \ .$$

If in addition both W and Y are positive semidefinite (or negative semidefinite), then

$$\min \|\delta A\|_2 \le \frac{1}{\max\ (\xi, (\omega\eta)^{1/2})} \ ;$$

this upper bound is within a factor of 2 of the lower bound.

Note: The upper bound does not apply to the matrix in (4.4).

Proof: To first order,

$$(T + \delta T)^{-1} = T^{-1} - T^{-1}\delta T T^{-1} = \begin{bmatrix} W & X^T \\ X & Y \end{bmatrix} - \begin{bmatrix} W\delta A^T X + X^T \delta A W & W\delta A^T Y + X^T \delta A X^T \\ X\delta A^T X + Y\delta A W & X\delta A^T Y + Y\delta A X^T \end{bmatrix}$$

Thus, to first order the perturbations in ω, ξ and η are bounded by $2\omega\xi\|\delta A\|_2$, $(\xi^2 + \omega\eta)\|\delta A\|_2$ and $2\eta\xi\|\delta A\|_2$. Let $\kappa = \xi + (\omega\eta)^{1/2}$. To first order the perturbation to κ is bounded by

$$\delta\kappa \le (\xi^2 + \omega\eta + 2\xi(\omega\eta)^{1/2})\|\delta A\|_2 = \kappa^2 \ \|\delta A\|_2 \ .$$

Note that $\|T^{-1}\|$ cannot become infinite unless κ becomes infinite. By Lemma 1, the smallest $\|\delta A\|_2$ such that $T + \delta T$ is singular is satisfies

$$\|\delta A\|_2 \ge \frac{1}{\kappa}$$

proving the lower bound.

To prove the upper bound we will consider 2 cases, when $\alpha = \xi/(\omega\eta)^{1/2} \le 1$, and when $\alpha > 1$. We assume without loss of generality that both W and Y are positive semidefinite (otherwise, negate T). Consider first the case $\alpha \le 1$. We will exhibit a δA whose norm satisfies the upper bound. Let u_ω and v_η be unit vectors such that $u_\omega^T W u_\omega = \omega$ and $v_\eta^T Y v_\eta = \eta$. Let $u = u_\omega / \omega^{1/2}$ and $v = \rho v_\eta / \eta^{1/2}$. Let $\delta A = vu^T$ and

$$V = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \ \text{and} \ U^T = \begin{bmatrix} 0 & v^T \\ u^T & 0 \end{bmatrix} \ ;$$

then $\delta T = VU^T$ and again by the Sherman-Morrison-Woodbury formula $T + \delta T$ is singular if and only if $I + U^T T^{-1} V$ is singular, or

$$(1 + v^T X u)^2 - (u^T W u)\cdot(v^T Y v) = 0 \ . \tag{4.5}$$

With our choice of u and v we may write $v^T X u = \rho\beta$ where $|\beta| \le \alpha$, as well as $u^T W u = 1$ and $v^T Y v = \rho^2$, so that (4.5) becomes

$$(1 + \rho\beta)^2 - \rho^2 = 0$$

or $\rho_\pm = -(\beta \pm 1)^{-1}$ (if $|\beta| = 1$ there is just one solution $|\rho| = .5$). Clearly $\min |\rho_\pm| \le 1$, so $\|\delta A\|_2 = \|u\|_2 \cdot \|v\|_2 \le (\omega\eta)^{-1/2}$, yielding the upper bound in the case $\alpha \le 1$.

When $\alpha > 1$, we need to choose u and v differently. Let u_ξ and v_ξ be unit vectors such that $v_\xi^T X u_\xi = \xi$, and let $u = \rho u_\xi / \xi$ and $v = v_\xi$. Note $\|\delta A\|_2 = |\rho|/\xi$. With this choice of u and v

we may write $(u^TWu)\cdot(v^TYv)=\beta\rho^2$, where $0\le\beta\le\alpha^{-2}<1$. Thus (4.5) becomes

$$(1+\rho)^2-\beta\rho^2 = 0$$

which has solutions $\rho_\pm = -(1\pm\beta^{1/2})^{-1}$. Since $\min|\rho_\pm| \le 1$, $||\delta A||_2 \le \xi^{-1}$, yielding the upper bound. □

Note that it is always possible to find a complex perturbation δA satisfying the upper bound, independent of the signs of the eigenvalues of H and M; this follows directly from the proof. However, if we change the problem to finding the smallest complex perturbation δA such that the Hermitian matrix

$$\begin{bmatrix} H & A^* \\ A & M \end{bmatrix}$$

is singular, we get the same necessary and sufficient conditions on the eigenvalues of H and M for the existence of δA as when everything is real.

As in Theorem 2, the lower bound for the 2-norm is a lower bound for the Frobenius norm, and the upper bound for the 2-norm is also an upper bound for the Frobenius norm because the perturbation attaining the upper bound has rank 1.

When H is positive (or negative) definite in (4.4), it is easy to see that the matrix there is invertible if and only if A has full rank, and therefore the smallest rank lowering perturbation of the matrix is simply the smallest rank lowering perturbation of A, and has norm $\sigma_{\min}(A)$. When A is indefinite $\sigma_{\min}(A)$ is still an upper bound on the norm of the smallest rank lowering perturbation, although it is no longer a lower bound as well.

In the next section we will extend Theorem 3 to permit perturbations in H as well as A.

5. The Smallest Rank Lowering Perturbation of Three Subblocks

In this section we are interested in the smallest $||\delta T||$ such that $T+\delta T$ is singular where

$$\delta T = \begin{bmatrix} \delta A & \delta B \\ 0 & \delta D \end{bmatrix} .$$

Here $||\cdot||$ may be any operator norm subject to the restrictions

$$\max(||E||, ||F||) = \left|\left|\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\right|\right| , \quad \left|\left|\begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}\right|\right| = ||G|| ,$$

and that the norm of a matrix is at least as large as the norm of any subblock. These restrictions are satisfied, for example, by all the matrix p-norms. As before, let

$$T^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

and $\omega=||W||$, $\xi=||X||$, $\eta=||Y||$, and $\zeta=||Z||$.

Theorem 4: Let $\kappa = \max(\zeta,\omega,\eta,(\xi\eta)^{1/2})$. Then the smallest $||\delta T|| = \left|\left|\begin{bmatrix} \delta A & \delta B \\ 0 & \delta D \end{bmatrix}\right|\right|$ such that

$$\begin{bmatrix} A+\delta A & B+\delta B \\ C & D+\delta D \end{bmatrix}$$

is singular is bounded by

$$\frac{1}{3\kappa} \le \min \|\delta T\| \le \frac{\gamma}{\kappa} ,$$

where $\gamma \le 3^{1/2}$. More specifically, if we define $\alpha = \max(\omega,\zeta,\eta) \,/\, (\xi\eta)^{1/2}$ then

$$\gamma = \begin{cases} 1 & \text{if } \alpha \ge 1 \\ \dfrac{1}{\alpha} & \text{if } 3^{-1/2} < \alpha < 1 \\ \dfrac{(1+\alpha^2)^{1/2}}{1-\alpha^2} & \text{if } \alpha \le 3^{-1/2} \end{cases} \tag{5.1}$$

If T and δT are real, this result can be improved to

$$\frac{1}{3\kappa} \le \min \|\delta T\| \le \frac{1}{\kappa} .$$

Proof: To first order,

$$(T+\delta T)^{-1} = T^{-1} - T^{-1}\delta T T^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} - \begin{bmatrix} W\delta AW + W\delta BY + X\delta DY & W\delta AX + W\delta BZ + X\delta DZ \\ Y\delta AW + Y\delta BY + Z\delta DY & Y\delta AX + Y\delta BZ + Z\delta DZ \end{bmatrix}$$

Thus, to first order the perturbations in ω, ξ, η and ζ are bounded by $(\omega^2+\omega\eta+\xi\eta)\|\delta T\|$, $(\omega\xi+\omega\zeta+\xi\zeta)\|\delta T\|$, $(\eta\omega+\eta^2+\zeta\eta)\|\delta T\|$, and $(\eta\xi+\eta\zeta+\zeta^2)\|\delta T\|$. Therefore, to first order the perturbation to κ is bounded by

$$\delta\kappa \le \|\delta T\| \cdot \begin{cases} \omega^2+\omega\eta+\xi\eta & \text{if } \omega=\kappa \\ \eta\xi+\eta\zeta+\zeta^2 & \text{if } \zeta=\kappa \\ \eta\omega+\eta^2+\zeta\eta & \text{if } \eta=\kappa \\ .5\cdot(\xi\eta)^{1/2}\cdot(2\omega+2\zeta+\eta) \;+\; .5(\eta/\xi)^{1/2}\omega\zeta & \text{if } (\xi\eta)^{1/2}=\kappa \end{cases}$$

as is easily verified by computation. Note that $\|T^{-1}\|$ cannot become infinite unless κ becomes infinite. The only term in $\delta\kappa$ that prevents us from immediately concluding that $\delta\kappa \le 3\kappa^2 \|\delta T\|$ is $.5(\eta/\xi)^{1/2}\omega\zeta$ if $(\xi\eta)^{1/2}=\kappa$. But $(\xi\eta)^{1/2}=\kappa$ implies $(\xi\eta)^{1/2}\ge\eta$ or $\xi\ge\eta$; thus $(\eta/\xi)^{1/2}\le 1$ so $\delta\kappa \le 3\kappa^2 \|\delta T\|$. From Lemma 1 we conclude $\|\delta T\| \ge 1/(3\kappa)$ if $T+\delta T$ is singular, proving the lower bound.

To prove the upper bound we will exhibit perturbations δT which attain it. The three cases indicated in (5.1) are essentially the same as in (4.1) of Theorem 2: In case 1, when $\alpha = \max(\omega,\zeta,\eta) \,/\, (\xi\eta)^{1/2} \ge 1$, we know from Theorem 1 that there is a perturbation of A alone (if ω is largest), B alone (if η is largest) or D alone (if ζ is largest) of norm $1/\kappa = 1/\max(\omega,\zeta,\eta)$ which makes $T+\delta T$ singular. In case 2, when $1>\alpha>3^{-1/2}$, the same perturbation of A, B or D alone has norm $1/\max(\omega,\zeta,\eta) = \dfrac{1}{\alpha\kappa}$.

In case 3, when $\alpha \le 3^{-1/2}$, we perturb δA and δD just as in the proof of Theorem 1, leaving $\delta B=0$. This completes the proof. □

It is interesting that it is never necessary to perturb all three entries A, B and D simultaneously to prove the upper bound in the theorem. Note also that the lower bound for the 2-norm $\|\cdot\|_2$ is a lower bound for the Frobenius norm $\|\cdot\|_F$, and that $2^{1/2}$ times the upper bound for the 2-norm is an upper bound for the Frobenius norm, since the perturbations attaining the upper bound have rank at most 2.

Now we return to the problem considered in the last section: Let

$$T = \begin{bmatrix} H & A^T \\ A & M \end{bmatrix} , \quad T^{-1} = \begin{bmatrix} W & X^T \\ X & Y \end{bmatrix} \quad \text{and} \quad \delta T = \begin{bmatrix} \delta H & \delta A^T \\ \delta A & 0 \end{bmatrix} \tag{5.2}$$

where all matrices are real and T and δT are symmetric. We seek $\min\|\delta T\|_2$ where $T+\delta T$ is singular.

Theorem 5: Let T, T^{-1} and δT be real matrices as in (5.2). Let $\omega = \|W\|_2$, $\xi = \|X\|_2$, $\eta = \|Y\|_2$, and $\kappa = \max(\omega, \xi, (\omega\eta)^{1/2})$. The smallest $\|\delta T\|_2$ that makes $T + \delta T$ singular satisfies

$$\frac{1}{3\kappa} \le \min \|\delta T\|_2 \ .$$

If in addition either $\kappa = \omega$ or both W and Y are positive semidefinite (or negative semidefinite), then

$$\min \|\delta T\|_2 \le \frac{1}{\kappa} \ .$$

Proof: To first order

$$(T+\delta T)^{-1} = T^{-1} - T^{-1}\delta T T^{-1}$$

$$= \begin{bmatrix} W & X^T \\ X & Y \end{bmatrix} - \begin{bmatrix} W\delta HW + W\delta A^T X + X^T \delta A W & W\delta HX^T + W\delta A^T Y + X^T \delta A X^T \\ X\delta HW + X\delta A^T X + Y\delta A W & X\delta H X^T + X\delta A^T Y + Y\delta A X^T \end{bmatrix} .$$

Thus, to first order the perturbations in ω, ξ and η are bounded by $(\omega^2 + 2\omega\xi)\|\delta T\|_2$, $(\xi^2 + \omega\xi + \omega\eta)\|\delta T\|_2$ and $(\xi^2 + 2\xi\eta)\|\delta T\|_2$. Thus, to first order the perturbation in κ is bounded by

$$\delta\kappa \le \|\delta T\|_2 \cdot \begin{cases} \omega^2 + 2\omega\xi & \text{if } \omega = \kappa \\ \xi^2 + \omega\xi + \omega\eta & \text{if } \xi = \kappa \\ 2\xi(\omega\eta)^{1/2} + .5\omega(\omega\eta)^{1/2} + .5\xi^2(\omega/\eta)^{1/2} & \text{if } (\omega\eta)^{1/2} = \kappa \end{cases}$$

Note that $\|T^{-1}\|$ can only become infinite if κ becomes infinite. The only term in $\delta\kappa$ that seems to prevent us from concluding that $\delta\kappa \le 3\kappa^2\|\delta T\|$ is $.5\xi^2(\omega/\eta)^{1/2}$ when $\kappa = (\omega\eta)^{1/2}$. But $\kappa = (\omega\eta)^{1/2}$ means $(\omega\eta)^{1/2} \ge \omega$ or $\eta \ge \omega$, whence $\omega/\eta \le 1$ and $\delta\kappa \le 3\kappa^2\|\delta T\|_2$. From Lemma 1 we conclude $\min\|\delta T\|_2 \ge 1/(3\kappa)$ as desired.

For the upper bound, there are two cases, depending on whether $\kappa = \omega$, or both W and Y are positive semidefinite (or negative semidefinite). In the first case Theorem 1 guarantees the existence of a symmetric perturbation of H alone with $\delta A = 0$ and $\|\delta H\|_2 = 1/\omega = 1/\kappa$ such that $T + \delta T$ is singular. In the second case Theorem 3 guarantees the existence of a perturbation of A alone with $\delta H = 0$ satisfying the upper bound. □

It is interesting that it is never necessary to perturb A and H simultaneously to prove the upper bound in the theorem. As with the Theorem 3, the upper and lower bounds for the 2-norm are also bounds for the Frobenius norm.

6. Extensions to 3 by 3 Block Matrices

In this section we discuss various smallest rank lowering structured perturbations δT of T where T and δT are 3 by 3 block matrices. Let

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{bmatrix} \tag{6.1}$$

where T and T^{-T} are conformally partitioned (i.e. T_{ij} and T^{ji} have the same dimensions). Our upper and lower bounds will be in terms of the $\tau_{ij} = \|T^{ij}\|$. These results include the cases where the last block row and/or column of T is null.

Theorems 2 through 5 extend to this case with essentially no changes in their proofs. In Theorems 6 to 9 it is possible for no rank lowering perturbation δT to exist. In Theorems 6 and 8 the lower and upper bounds will both be infinite in this situation. If the lower bound in Theorems 7 and 9 is infinite this also implies no rank lowering perturbation exists. Unfortunately, the extension of this approach to perturbations of all diagonal blocks T_{ii} is not as complete, and it is possible for a large gap to exist between the upper and lower bounds.

Theorem 6: Let T and T^{-1} be as in (6.1), and $\tau_{ij} = \|T^{ij}\|$. Let

$$\delta T = \begin{bmatrix} \delta T_{11} & 0 & 0 \\ 0 & \delta T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be conformally partitioned with T. Let the norm $\|\cdot\|$ satisfy $\|\mathrm{diag}(X,Y)\| = \max(\|X\|,\|Y\|)$. Let $\kappa = \max(\tau_{11},\tau_{22},(\tau_{12}\tau_{21})^{1/2})$. Then the smallest $\|\delta T\|$ such that $T+\delta T$ is singular satisfies

$$\frac{1}{2\kappa} \le \min \|\mathrm{diag}(\delta A,\delta B)\| \le \frac{\gamma}{\kappa} \ ,$$

where $\gamma \le 3^{1/2}$. More specifically, if we define $\alpha = \max(\tau_{11},\tau_{22}) \,/\, (\tau_{12}\tau_{21})^{1/2}$ then

$$\gamma = \begin{cases} 1 & \text{if } \alpha \ge 1 \\ \dfrac{1}{\alpha} & \text{if } 3^{-1/2} < \alpha < 1 \\ \dfrac{(1+\alpha^2)^{1/2}}{1-\alpha^2} & \text{if } \alpha \le 3^{-1/2} \end{cases}$$

If T is real, this result may be improved to

$$\frac{1}{\max(\tau_{11},\tau_{22}) + (\tau_{12}\tau_{21})^{1/2}} \le \min \|\mathrm{diag}(\delta A,\delta B)\| \le \frac{1}{\max(\tau_{11},\tau_{22},(\tau_{12}\tau_{21})^{1/2})} \ ;$$

note that the upper bound is at most twice the lower bound. If T is real symmetric (complex Hermitian) and the norm is the 2-norm $\|\cdot\|_2$, then there are δA and δD satisfying the upper bound of the theorem which are also real symmetric (complex Hermitian).

Sketch of Proof: The proof is essentially identical to that of Theorem 2. The only difference is that the rank 2 matrices

$$V = \begin{bmatrix} v_A & 0 \\ 0 & v_D \end{bmatrix} \text{ and } U^T = \begin{bmatrix} u_A^T & 0 \\ 0 & u_D^T \end{bmatrix}$$

whose product VU^T equaled δT in the proof of Theorem 2 must be enlarged to

$$V = \begin{bmatrix} v_A & 0 \\ 0 & v_D \\ 0 & 0 \end{bmatrix} \text{ and } U^T = \begin{bmatrix} u_A^T & 0 & 0 \\ 0 & u_D^T & 0 \end{bmatrix} .$$

Theorem 7: Let T and T^{-1} in (6.1) be real and symmetric, and let the partitioning be symmetric as well ($T_{ij} = T_{ji}^T$). Let $\tau_{ij} = \|T^{ij}\|_2 = \tau_{ji}$. Let

$$\delta T = \begin{bmatrix} 0 & \delta T_{21}^T & 0 \\ \delta T_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be conformally partitioned with T. The smallest $\|\delta T\|_2$ such that $T+\delta T$ is singular satisfies

$$\frac{1}{\tau_{21}+(\tau_{11}\tau_{22})^{1/2}} \le \min\|\delta T\|_2 \ .$$

If in addition both T^{11} and T^{22} are positive semidefinite (or negative semidefinite), then

$$\min\|\delta T\|_2 \le \frac{1}{\max(\tau_{21},(\tau_{11}\tau_{22})^{1/2})} \ .$$

Sketch of Proof: The proof of Theorem 3 is essentially identical, except for the same minor change required for the proof of Theorem 6.

Theorem 8: Let T and T^{-1} be as in (6.1), and $\tau_{ij} = \|T^{ij}\|$, where the norm satisfies the same conditions as in Theorem 4. Let

$$\delta T = \begin{bmatrix} \delta T_{11} & \delta T_{12} & 0 \\ 0 & \delta T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be conformally partitioned with T. Let $\kappa = \max(\tau_{11},\tau_{22},\tau_{21},(\tau_{12}\tau_{21})^{1/2})$. Then the smallest $\|\delta T\|$ such that $T+\delta T$ is singular is bounded by

$$\frac{1}{3\kappa} \le \min\|\delta T\| \le \frac{\gamma}{\kappa}$$

where $\gamma \le 3^{1/2}$. More specifically, if we define $\alpha = \max(\tau_{11},\tau_{22},\tau_{21}) \ / \ (\tau_{12}\tau_{21})^{1/2}$ then

$$\gamma = \begin{cases} 1 & \text{if } \alpha \ge 1 \\ \dfrac{1}{\alpha} & \text{if } 3^{-1/2} < \alpha < 1 \\ \dfrac{(1+\alpha^2)^{1/2}}{1-\alpha^2} & \text{if } \alpha \le 3^{-1/2} \end{cases} \tag{5.1}$$

If T and δT are real, this result may be improved to

$$\frac{1}{3\kappa} \le \min\|\delta T\| \le \frac{1}{\kappa} \ .$$

Again, with only slight modification the proof of Theorem 4 goes through for these results as well.

Theorem 9: Let T and T^{-1} be real symmetric matrices as in (6.1), where the partitioning is also symmetric so $T_{ij} = T_{ji}^T$ and $\tau_{ij} = \|T^{ij}\|_2 = \tau_{ji}$. Let

$$\delta T = \begin{bmatrix} \delta T_{11} & \delta T_{21}^T & 0 \\ \delta T_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be partitioned conformally with T. Let $\kappa = \max(\tau_{11},\tau_{21},(\tau_{11}\tau_{22})^{1/2})$. Then the smallest $\|\delta T\|_2$ such that $T+\delta T$ is singular is bounded by

$$\frac{1}{3\kappa} \le \min\|\delta T\|_2 \ .$$

If in addition either $\kappa=\tau_{11}$ or both T^{11} and T^{22} are positive semidefinite (negative semidefinite), then

$$\min\|\delta T\|_2 \le \frac{1}{\kappa} \ .$$

Again, the proof is a minor modification of the proof of Theorem 5.

Finally, we consider the block diagonal perturbation

$$\delta T = \begin{bmatrix} \delta T_{11} & 0 & 0 \\ 0 & \delta T_{22} & 0 \\ 0 & 0 & \delta T_{33} \end{bmatrix} \qquad (6.2)$$

where δT is conformally partitioned to T. This time, our lower bound technique based on Lemma 1 yields a lower bound that is not always attainable:

Theorem 10: Let T, T^{-1} and δT be as in (6.1) and (6.2). Let $\tau_{ij} = \|T^{ij}\|$, where $\|\mathrm{diag}(X,Y)\| = \max(\|X\|,\|Y\|)$. Let

$$\kappa_1 = \max\,(\tau_{11},\tau_{22},\tau_{33},(\tau_{12}\tau_{21})^{1/2},(\tau_{13}\tau_{31})^{1/2},(\tau_{23}\tau_{32})^{1/2})$$

and

$$\kappa_2 = \max\,(\frac{\tau_{23}\tau_{31}}{\tau_{21}},\frac{\tau_{13}\tau_{32}}{\tau_{12}},\frac{\tau_{32}\tau_{21}}{\tau_{31}},\frac{\tau_{12}\tau_{23}}{\tau_{13}},\frac{\tau_{31}\tau_{12}}{\tau_{32}},\frac{\tau_{21}\tau_{13}}{\tau_{23}}) = \max_{i,j,k \text{ distinct}} (\frac{\tau_{ij}\tau_{jk}}{\tau_{ik}}) \; .$$

Then the smallest $\|\delta T\|$ such that $T + \delta T$ is singular is bounded by

$$\frac{1}{9\max\,(\kappa_1,\kappa_2)} \le \min \|\delta T\| \le \frac{\gamma}{\kappa_1}$$

where $\gamma \le 3^{1/2}$. Note that if $\kappa_2 >> \kappa_1$, the lower bound will be much smaller than the upper bound.

Proof: The upper bound comes from Theorem 6 and considering perturbations to any two of the δT_{ii} alone. The lower bound comes from applying Lemma 1 to $\kappa = \max(\kappa_1,\kappa_2)$: To first order $(T+\delta T)^{-1} = T^{-1} - T^{-1}\delta T T^{-1}$. From $T^{-1}\delta T T^{-1}$ we get first order bounds on the perturbations of all the terms in κ, and show that to first order the change in κ is bounded by

$$\delta\kappa \le 9\kappa^2 \, \|\delta T\| \; .$$

Using Lemma 1, we conclude $\min \|\delta T\| \ge 1/(9\kappa)$. □

If some τ_{ij}, $i \ne j$ is small or zero, then κ_2 may be very large or infinite, and the lower bound will be much too small, even smaller than the trivial lower bound $\|T^{-1}\|^{-1}$. However, it is sometimes close to the upper bound (and larger than $\|T^{-1}\|^{-1}$), and so supplies useful information, as in the following 3 by 3 example:

$$T^{-1} = \begin{bmatrix} 1 & x^2 & x \\ 1 & 1 & 1 \\ 1 & x & 1 \end{bmatrix}$$

where $x >> 1$. In this example $\kappa_1 = \kappa_2 = x \ll x^2 \approx \|T^{-1}\|$.

Just as in Theorems 6 through 9, Theorem 10 can be extended to the case of perturbing three diagonal subblocks of a block 4 by 4 matrix.

Note that for Theorems 6, 8 and 10, the lower bound for the 2-norm $\|\cdot\|_2$ is also a lower bound for the Frobenius norm $\|\cdot\|_F$, and that $2^{1/2}$ times the upper bound for the 2-norm is an upper bound for the Frobenius norm, since the perturbation attaining the upper bound is at most rank 2. In Theorems 7 and 9, the upper and lower bounds for the 2-norm are also upper and lower bounds for the Frobenius norm.

7. Estimating Norms of Sublocks of T^{-1}

The estimates in this paper depend on the norms of sublocks of T^{-1}. It therefore appears that computing these bounds is as expensive as computing the inverse of T. However, provided it is inexpensive to solve $Tx=b$ and $T^Ty=c$ for various right hand sides b and c (using, for example, an existing LU factorization of T), it is not much more expensive to estimate the norm of any subblock. Therefore, the bounds in this paper may be computed in practice even for large sparse matrices whose inverses would be dense, as long as we can solve linear systems efficiently.

Our technique depends on an algorithm of Hager [Hager, Higham] which can estimate the 1-norm or ∞-norm of a matrix B given the ability to compute Bx and B^Ty for any x and y. Suppose without loss of generality that the submatrix of T^{-1} whose 1-norm or ∞-norm we want to estimate lies in the leading r rows and c columns of T^{-1}. Let E_r be a diagonal matrix with 1's in diagonal positions 1 through r and zeros elsewhere, and define E_c analogously. Then what we want to estimate is $||E_rT^{-1}E_c||$. Using Hager's method, we only need to be able to multiply any vector by either $E_rT^{-1}E_c$ or $E_cT^{-T}E_r$; both multiplications are inexpensive given an LU factorization of T or other inexpensive method for solving $Tx=b$ and $T^Ty=c$.

Hager's algorithm also supplies vectors u and v^T such that v^TBu is nearly as large as possible. These vectors can be used to construct perturbations nearly satisfying the upper bounds in the previous sections, or as starting vectors in iterations designed to compute the smallest perturbations more accurately.

8. Conclusions and Future Work

We have shown how to estimate the norm of the smallest block structured additive perturbation of a block 2 by 2 matrix that makes it singular. Our estimates are accurate to within a factor of at most $3^{3/2}$ (a factor of 3 for real matrices), and work for all possible block perturbations of a block 2 by 2 matrix and for a large class of matrix norms, including all p-norms and the Frobenius norm. Using an algorithm of Hager, we may inexpensively estimate our bounds even for large matrices. We explicitly exhibit rank one or rank two perturbations which achieve our upper bounds. These explicit perturbations may be used as starting values for an optimization routine designed to compute the answer to higher accuracy than our a priori estimates provide. These results extend to some block perturbations of 3 by 3 block matrices, although the upper and lower bounds may not always be close.

Our technique, based on differential inequalities introduced in [Demmel87a], can be applied to other block perturbations not discussed in this paper. It is an open question as to whether it can be used to get tight upper and lower bounds for more complicated structures than those addressed here, as well as for nonsquare matrices; this is future work.

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